Publ. Mat. **57** (2013), 139–153
DOI: 10.5565/PUBLMAT_57113_06

ON THE FIXED-POINT SET OF AN AUTOMORPHISM OF A GROUP

B. A. F. Wehrfritz

Abstract: Let ϕ be an automorphism of a group G. Under various finiteness or solubility hypotheses, for example under polycyclicity, the commutator subgroup $[G,\phi]$ has finite index in G if the fixed-point set $C_G(\phi)$ of ϕ in G is finite, but not conversely, even for polycyclic groups G. Here we consider a stronger, yet natural, notion of what it means for $[G,\phi]$ to have 'finite index' in G and show that in many situations, including G polycyclic, it is equivalent to $C_G(\phi)$ being finite.

2010 Mathematics Subject Classification: 20F16, 20E36.

Key words: automorphism, fixed-point set, soluble group.

Endimioni and Moravec in [**2**] prove that if ϕ is an automorphism of a polycyclic group G with its fixed-point set $C_G(\phi)$ finite, then G modulo $[G,\phi] = \langle g^{-1}\cdot g\phi : g \in G\rangle$ is also finite (so $[G,\phi]$ is large), but the converse is false, even if ϕ has order 2, $C_G(\phi) = \langle 1\rangle$ and G is polycyclic and metabelian. These and related results are extended in [**9**] to, in particular, soluble groups of finite rank. Is there some stronger notion of $[G,\phi]$ being large such that in these situations $C_G(\phi)$ is finite if and only is $[G,\phi]$ is large in this sense? We will see below that the answer to this is yes.

Let ϕ be an automorphism of a group G and define the map γ if G into itself by $g\gamma = [g,\phi] = g^{-1}\cdot g\phi$. Then $\ker\gamma = \{g \in G : g\gamma = 1\}$ is $C_G(\phi)$ and assumptions on $C_G(\phi)$ should give information about $G\gamma$. The problem is that γ is not usually a homomorphism and $G\gamma$ is not usually a subgroup of G. If S is any subset of G say that S has finite index in G if S contains a subgroup of G (normal if you wish) of finite index in G. If $G\gamma$ has finite index in G then so does $[G,\phi]$, since $[G,\phi] = \langle G\gamma\rangle$. We shall see that in suitable situations $C_G(\phi)$ is finite if and only if $G\gamma$ has finite index in G, and these situations include polycyclic groups and soluble groups of finite rank.

We start by defining the classes of group we shall be mainly considering. A group G has finite Hirsch number if it has a series of finite length, each factor of which is either locally finite or infinite cyclic, the number of infinite cycles here being an invariant of G, called the Hirsch number of G. For any set π of primes a group G satisfies min-π (min-q if $\pi = \{q\}$) if the set of π-subgroups of G satisfies the minimal condition under inclusion.

An FAR group (that is, a finite-abelian-ranks group, also called an $\mathbf{S}_0$-group) is a soluble group with finite Hirsch number satisfying min-q for every prime q. See [**5**] and [**6**] for discussions and alternative definitions of this class of FAR groups. Note in particular that all soluble groups of finite (Prüfer) rank and hence all polycyclic groups are FAR groups.

Theorem 1. *Let G be a finite extension of a soluble FAR group and ϕ an automorphism of G such that $\phi^p = 1$ for some prime p. Define maps γ and ψ of G into itself by*

$$g\gamma = g^{-1} \cdot g\phi \quad \textit{and} \quad g\psi = g \cdot g\phi \cdot g\phi^2 \cdots g\phi^{p-1}.$$

The following are equivalent.

a) $C_G(\phi)$ *is finite.*

b) $G\gamma$ *has finite index in* G.

c) $\ker\psi$ *has finite index in* G.

Here $\ker\psi = \{g \in G : g\psi = 1\}$. Again note that ψ is not usually a homomorphism and $\ker\psi$ is not usually a subgroup of G; also always $G\gamma\psi = \{1\}$, so $\ker\psi \supseteq G\gamma$. The main part of the theorem is the equivalence of a) and b). The equivalence of a) and c) just for polycyclic-by-finite groups is almost immediate from Corollary C of [**1**]. Further Theorem B of [**1**] implies that in Theorem 1 above $C_G(\phi)$ being finite is equivalent to $\ker(\psi^2)$ having finite index in G.

Sometimes one can deduce that $G/[G, \phi]$ is periodic from the periodicity of $C_G(\phi)$, e.g. see Lemma 1 of [**11**]. If S is a subset of a group G, say that G is periodic modulo S if for each $g \in G$ there is a positive integer n with $g^n \in S$. Note that this is weaker than requiring S to contain a normal subgroup N of G with G/N periodic, even in the simplest of cases. For example, let G denote the additive group of the integers $\mathbf{Z}$ and $S = \{\pm ip_i : i = 0, 1, 2, \dots\}$, where p_i denotes the $(i + 1)$-th prime. Trivially G is periodic modulo S, but if $s \in S$ with $s \neq 0$ it is easy to see that $2s \notin S$, so S contains no non-zero subgroup of G. Also say G has finite exponent modulo S if for some integer $e > 0$, $g^e \in S$ for all $g \in G$.

Theorem 2. *Let G be a finite extension of a soluble FAR group and ϕ an automorphism of G such that $\phi^p = 1$ for some prime p. Define maps γ and ψ of G into itself by*

$$g\gamma = g^{-1} \cdot g\phi \quad \text{and} \quad g\psi = g \cdot g\phi \cdot g\phi^2 \cdots g\phi^{p-1}.$$

The following are equivalent.

a) *$C_G(\phi)$ is periodic.*

b) *There is a finitely generated nilpotent subgroup N of G contained in $G\gamma$ such that G is periodic modulo N.*

c) *G is periodic modulo $G\gamma$.*

d) *G is periodic modulo* $\ker\psi$.

We can squeeze a little more out of this theorem.

Corollary. *Let G be a group that has a local system consisting of finite extensions of soluble FAR groups. Suppose ϕ is an automorphism of G satisfying $\phi^p = 1$ for some prime p. With γ and ψ as in Theorem 2, the following are equivalent.*

a) *$C_G(\phi)$ is periodic.*

b) *G is periodic modulo $G\gamma$.*

c) *G is periodic modulo* $\ker\psi$.

So, suppose in Theorem 2 (or its corollary) that G is a (torsion-free)-by-finite FAR group. If $C_G(\phi)$ is periodic, then $C_G(\phi)$ is finite. Hence if G is periodic modulo $G\gamma$ (resp. $\ker\psi$), then $C_G(\phi)$ is finite by Theorem 2 and consequently by Theorem 1 the subset $G\gamma$ (resp. $\ker\psi$) has finite index in G. Hence in particular if G is polycyclic-by- finite and G is periodic modulo $G\gamma$ or $\ker\psi$, then G has a normal subgroup N with $\ker\psi \supseteq G\gamma \supseteq N$ and G/N periodic (i.e. finite). This does not hold for soluble FAR groups in general, indeed not even for minimax such groups.

Example 1. There is a metabelian minimax group G of rank 3 with an automorphism ϕ of order 2 such that $C_G(\phi)$ is periodic and yet $G\gamma$ and $\ker\psi$ do not contain normal subgroups N of G such that G/N is periodic.

If ϕ has infinite order then we can no longer define ψ, but at least we can say something about $G\gamma$.

Theorem 3. *Let G be a finite extension of a nilpotent FAR group and let ϕ be an automorphism of G with γ defined as in the previous theorems.*

a) *If $C_G(\phi)$ is finite, then $G\gamma$ has finite index in G.*

b) *If ϕ has finite order, say $\phi^m = 1$, with $g\psi = g \cdot g\phi \cdot g\phi^2 \cdots g\phi^{m-1}$ as usual, then the following are equivalent.*

   i) *$C_G(\phi)$ is finite.*

   ii) *$G\gamma$ has finite index in G.*

   iii) *$\ker\psi$ has finite index in G.*

Theorem 4. *Let G be a finite extension of a soluble FAR group and ϕ an automorphism of G with $C_G(\phi)$ finite. Then with γ as above*

$$(G\gamma)^{[3]} = \{x\gamma \cdot y\gamma \cdot z\gamma : x, y, z \in G\}$$

has finite index in G.

Theorem 4 strengthens Part ii) of the theorem of [**9**], which says that $G/[G,\phi]$ is finite. Throughout this paper if ϕ is an automorphism of a group G and m is a positive integer with $\phi^m = 1$, we define maps γ and ψ of G into itself by

$$g\gamma = [g,\phi] = g^{-1} \cdot g\phi \quad \text{and} \quad g\psi = g \cdot g\phi \cdot g\phi^2 \cdots g\phi^{m-1}$$

for all $g \in G$. Notice that $G\gamma\psi = \{1\}$, so always $\ker\psi \supseteq G\gamma$. However $G\psi\gamma$ need not be $\{1\}$ in general. If $m = p$ is prime we write p for m in the definition of ψ. Also $\ker(\psi^r) \supseteq \ker\psi$ for every positive integer r.

Obviously the very much simpler case where the group G is abelian is basic to our results. In this case γ and ψ are endomorphisms, so $G\gamma = [G,\phi]$ and $\ker\psi$ are subgroups of G, $\ker\gamma = C_G(\phi)$ and $G/C_G(\phi) \cong G\gamma \leq G$. Thus if G is also finitely generated, then $C_G(\phi)$ is finite if and only if $G/[G,\phi]$ is finite. In general if ϕ has finite order then G has a local system of ϕ-invariant, finitely generated subgroups and a simple localization argument, together with a special case of Lemma 3 below shows that $C_G(\phi)$ is periodic if and only if $G/[G,\phi]$ is periodic.

For abelian groups in general we cannot replace periodic by finite here and this already just for abelian groups shows the necessity for some sort of rank restrictions at least in Theorem 1. If G is a free abelian group of infinite rank and ϕ is inversion on G, then $|\phi| = 2$, $C_G(\phi) = \langle 1\rangle$ and $G/[G,\phi]$ is an infinite elementary abelian 2-group. (The above already shows that if G is a torsion-free abelian group with $G/[G,\phi]$ periodic and $|\phi|$ finite, then $C_G(\phi) = \langle 1\rangle$.) If G is a direct product of infinitely many Prüfer 2-groups and again ϕ denotes inversion, then $|\phi| = 2$, $G/[G,\phi] = \langle 1\rangle$ and $C_G(\phi)$ is an infinite elementary abelian 2-group. (If G is periodic abelian with $C_G(\phi) = \langle 1\rangle$, then $G = [G,\phi]$, e.g. by [**3**, 10.1.1].)

Let A be the direct product of Prüfer q-groups A_q, one for each prime q, and let ϕ denote the automorphism of A of infinite order that for each q raises each element of A_q to its $(1 + q)$-th power. Then $A_q\gamma = (A_q)^q = A_q$ and so $[A, \phi] = A\gamma = A$. Also $C_A(\phi)$ is the direct product of cyclic groups of order q, one for each prime q, so $C_A(\phi)$ is infinite. Clearly A is an FAR group with rank 1 and Hirsch number 0. Consequently the converses of Theorem 3a) and Theorem 4 are both false.

However for special FAR groups it is possible to go further. Following the terminology of [**5**], a soluble group G is an FATR group (called an $\mathbf{S}_1$-group in [**6**]) if it is an FAR group and if the set of primes q, such that G contains an element of order q, is finite. In Proposition 3 below we prove that if ϕ is an automorphism of a finite extension of a soluble FATR group G with $G\gamma$ of finite index in G, then $C_G(\phi)$ is finite. Thus in particular, if in Theorem 3 we replace the FAR assumption by FATR, then $C_G(\phi)$ is finite if and only if $G\gamma$ has finite index in G. Theorem 4, of course, still leaves unanswered questions.

To prove Theorem 1 we need the following four lemmas. The first three are very elementary, but Lemma 4 is the heart of the proofs of Theorems 1 and 2 and, in parts, of Theorems 3 and 4. Lemma 1 is immediate from [**9**, Lemma 2].

Lemma 1. *Let ϕ be a fixed-point-free automorphism of the torsion-free abelian group A of finite rank. Then $A/A\gamma$ is finite.*

Lemma 2. *Let N be a normal subgroup of a group G lying in the n-th term $\zeta_n(G)$ of the upper central series of G for some $1 \leq n < \infty$. Set $N_i = N \cap \zeta_i(G)$ and suppose N_1 has finite exponent e. Then $(N_{i+1})^e \leq N_i$ for each i and N has exponent dividing e^n. If N also satisfies the minimal condition on subgroups, then N is finite.*

Proof: Let $x \in N_2$ and $g \in G$. Then $[x^e, g] = [x, g]^e = 1$. Thus $x^e \in \zeta_1(G)$, so $x^e \in N_1$ and $(N_2)^e \leq N_1$. A simple induction yields that $(N_{i+1})^e \leq N_i$ for each i and the remainder of the lemma follows easily. □

Lemma 3. *Let ϕ be an automorphism of finite order m of the group G. If G is periodic modulo* $\ker\psi$*, then $C_G(\phi)$ is periodic. If G has finite exponent e modulo* $\ker\psi$*, then $C_G(\phi)$ has finite exponent dividing em.*

Proof: For if $x \in C_G(\phi)$, then $x^n \in \ker\psi$ for some $n \geq 1$ and then $1 = x^n\psi = x^n \cdot x^n\phi \cdots x^n\phi^{m-1} = x^{mn}$. Hence $C_G(\phi)$ is periodic. If e is such that $g^e \in \ker\psi$ for all $g \in G$, then $x^{em} = 1$ and $C_G(\phi)$ has exponent dividing em. □

Lemma 4. *Let ϕ be an automorphism of finite order m of the nilpotent group G with $C = C_G(\phi)$ a finite π-group for some finite set π of primes. If m is a π-number and if G has finite Hirsch number and satisfies* min-q *for all $q \in \pi$ (equivalently* min-π *here), then $G\gamma$ has finite index in G.*

Proof: Note first (e.g. by [**4**, 3.13] and [**8**, Lemma 4]) that any image of a group with finite Hirsch number and min-q for some prime q also satisfies min-q. Assume by induction on the Hirsch number that the lemma is valid for groups (if any) of smaller Hirsch number than that of G.

Consider first the case where the maximum normal locally finite subgroup T of G is finite. If G has Hirsch number 0 then $G = T$ is finite in this case and the claim is vacuous. Suppose G has positive Hirsch number. Then G is (torsion-free)-by-finite (e.g. by [**8**, Lemmas 4 and 6]). Also G is periodic and hence finite if its centre $\zeta_1(G)$ is finite (e.g. by Lemma 2), so G has an infinite, ϕ-invariant, torsion-free central subgroup Z. Then $C_Z(\phi) = C \cap Z = \langle 1 \rangle$ and $(Z : Z\gamma)$ is finite by Lemma 1. Since $Z\gamma$ is infinite and torsion-free, the Hirsch number of $G/Z\gamma$ is less than that of G. Further by Lemma 1c) of [**10**] the order of $C_{G/Z\gamma}(\phi)$ divides $(Z\gamma : (Z\gamma)^m)|C|$, which is a finite π-number. By induction there exists a normal subgroup N of G of finite index with $Z\gamma \leq N$ and $N/Z\gamma$ lying in

$$(G/Z\gamma)\gamma = G\gamma \cdot Z\gamma/Z\gamma = G\gamma/Z\gamma,$$

since if $g \in G$ and $z \in Z$, then $g^{-1} \cdot g\phi \cdot z^{-1} \cdot z\phi = (gz)^{-1}(gz)\phi \in G\gamma$ by the centrality of z. Therefore $G\gamma \supseteq N$ and $G\gamma$ has finite index in G.

Now we consider the case where $O_\pi(G)$ is finite. We further induct on the least c such that $O_{\pi'}(G) \leq \zeta_c(G)$. Obviously c is bounded by the class of G and the case $c = 0$ is covered by the case completed above. Suppose $c \geq 1$ and set $Z = O_{\pi'}(G) \cap \zeta_1(G) = O_{\pi'}(\zeta_1(G))$. Then ϕ acts fixed-point freely on Z, so $Z = Z\gamma$, e.g. by [**3**, 10.1.1]. Again $C_{G/Z}(\phi)$ is a finite π-group by [**10**, Lemma 1c)] since by hypothesis no prime divisor of m lies in π', so $Z^m = Z$. Clearly

$$O_{\pi'}(G/Z) = O_{\pi'}(G)/Z \leq \zeta_{c-1}(G/Z).$$

By induction there is a normal subgroup N of G of finite index with $Z \leq N$ and with N/Z contained in $(G/Z)\gamma = G\gamma/Z$. Then $G\gamma \supseteq N$ and $G\gamma$ has finite index in G.

We now consider the general case. By hypothesis $O_\pi(G)$ satisfies the minimal condition on subgroups. Hence $O_\pi(G)$ is a Chernikov group and as such has a characteristic subgroup D of finite index that is a direct product of a finite number, say r, of Prüfer q-groups for the various primes q in π. Here we induct on r, the case $r = 0$ having been covered above.

Assume $r > 0$. By Lemma 2 the subgroup $Z = O_\pi(G) \cap \zeta_1(G)$ is infinite. Then Z contains a characteristic (in G) divisible subgroup E of finite index, $E\phi = E \leq D$, $E \neq \langle 1 \rangle$ and D/E is a direct product of less than r Prüfer groups. Clearly $E^m = E$ and then $C_{G/E}(\phi)$ is a finite π-group by [**10**, Lemma 1c)] again. By induction there is a normal subgroup N of G of finite index with $E \leq N$ and N/E contained in $(G/E)\gamma$. It follows that $G\gamma \supseteq N$ and that $G\gamma$ has finite index in G. The proof of the lemma is complete. □

Proposition 1. *Let ϕ be an automorphism of the group G with $\phi^p = 1$ for some prime p such that $C_G(\phi)$ is a finite π-group for some set π of primes. Suppose G has finite Hirsch number and satisfies* min-q *for all primes $q \in \pi$. Then $G\gamma$ has finite index in G.*

Proof: By Theorem D of [**1**] there is a ϕ-invariant nilpotent normal subgroup M of G of finite index. If $O_p(G) \neq \langle 1 \rangle$, then $O_p(G) \cap C_G(\phi) \neq \langle 1 \rangle$ and $p \in \pi$. Thus either way G satisfies min-p and we may assume that $p \in \pi$. By Lemma 4 there is a subgroup N of M of finite index with $M\gamma \supseteq N$. Clearly N has finite index in G and $G\gamma \supseteq M\gamma \supseteq N$. The proposition follows. □

The Proof of Theorem 1: a) implies b) by Proposition 1 and b) implies c) since $\ker \psi \supseteq G\gamma$. Finally if c) holds then $C_G(\phi)$ has finite exponent by Lemma 3. In particular $C_G(\phi)$ is a π-group for some finite set π of primes. For each prime q by hypothesis every q-subgroup of G is a Chernikov group and Chernikov groups of finite exponent are finite. Therefore $C_G(\phi)$ is finite. □

Example 2. We cannot remove the min-q condition completely from either Theorem 1 or Proposition 1.

Example 18 of [**1**] constructs a group G with the following properties. G is nilpotent of class 2 and prime exponent q and has an automorphism ϕ of prime order $p \neq q$ such that $C_G(\phi)$ has order q and $N\psi \neq \{1\}$ for all subgroups N of G of finite index. Clearly then neither $G\gamma$ nor $\ker \psi$ has finite index in G. Trivially G has Hirsch number 0 and satisfies min-r for all primes $r \neq q$. Thus we do need to restrict the r-subgroups of G at least for those primes r involved in $C_G(\phi)$. Moreover in this construction the prime p (but not the prime q) can be chosen arbitrarily.

We now come to the proof of Theorem 2. We need some further lemmas. Lemma 5 below is a special case of [**8**, Lemma 15].

Lemma 5. *Let ϕ be an automorphism of the group G with $\phi^p = 1$ for some prime p. Suppose A is an abelian ϕ-invariant normal p'-subgroup of G and set $C = C_G(\phi)$ and $K/A = C_{G/A}(\phi)$. Then $K = (A \cap K)C$. If C is also periodic, then so is K.*

Lemma 6. *Let ϕ be an automorphism of the group G with $\phi^p = 1$ for some prime p. Suppose A is an abelian ϕ-invariant normal divisible p-subgroup of G with finite rank r and set $C = C_G(\phi)$ and $K/A = C_{G/A}(\phi)$. Then $K^{p\cdot|\mathrm{GL}(r,p)|} \leq AC$. If C is also periodic, then so is K.*

Proof: Let $k \in K$. Then $k\gamma = a^p$ for some $a \in A$. Also $a^p \equiv a\psi$ modulo $A\gamma$, so $a^p = b\gamma \cdot c$ for some $b \in A$ and $c = a\psi \in C$. Hence $k\gamma = b\gamma \cdot c = b^{-1} \cdot b\phi \cdot c$ and

$$(kb^{-1})\gamma = bk^{-1} \cdot k\phi \cdot b^{-1}\phi = b \cdot k\gamma \cdot b^{-1}\phi = bb^{-1} \cdot b\phi \cdot c \cdot b^{-1}\phi = c \in A \cap C.$$

The actions of k and ϕ on A commute, so k and kb^{-1} normalize $A \cap C$. Then ϕ stabilizes the series $\langle kb^{-1}\rangle(A \cap C) \geq A \cap C \geq \langle 1\rangle$ and $\phi^p = 1$, so

$$c^p = [kb^{-1}, \phi]^p = [kb^{-1}, \phi^p] = 1.$$

Hence if $B = \{a \in A \cap C : a^p = 1\}$, then $c \in B$, ϕ stabilizes the series $\langle kb^{-1}\rangle B \geq B \geq \langle 1\rangle$ and B is a finite elementary abelian p-group of rank at most r normalized by kb^{-1}. Thus $(kb^{-1})^s$ centralizes B for $s = |\mathrm{GL}(r,p)|$ and consequently

$$[(kb^{-1})^{ps}, \phi] = [(kb^{-1})^s, \phi]^p = 1.$$

Therefore $(kb^{-1})^{ps} \in C$, $k^{ps} \in AC$, and $K^{ps} \leq AC$. Finally if C is periodic, then so is AC and therefore so is K. □

Lemma 7. *Let G be a nilpotent group with finite Hirsch number. Then G is periodic modulo one (and hence many) of its finitely generated subgroups.*

Proof: The upper central factors $\zeta_i(G)/\zeta_{i-1}(G)$ of G have finite torsion-free rank. Thus for each i there is a finite subset X_i of $\zeta_i(G)$ such that $\zeta_i(G)/\langle X_i\rangle\zeta_{i-1}(G)$ is periodic. Set $X = \bigcup_i X_i$, where i runs over the positive integers at most the class of G, and $N = \langle X\rangle$. We prove by induction on the class of G that G is periodic modulo N.

If G is abelian the claim is clear. We may assume by induction that G is periodic modulo $N\zeta_1(G)$. Then if $g \in G$ there exists $m \geq 1$ with $g^m \in N\zeta_1(G)$, say $g^m = yz$, where $y \in N$ and $z \in \zeta_1(G)$. Now $\zeta_1(G)/\langle X_1\rangle$ is periodic, so there exists $n \geq 1$ with $z^n \in \langle X_1\rangle \leq N$. Consequently $g^{mn} = y^n z^n \in N$. The lemma follows. □

Proposition 2. *Let G be a soluble-by-finite countable group with finite Hirsch number satisfying* min-p *for some prime p. Suppose G has an automorphism ϕ with $\phi^p = 1$. Then the following are equivalent.*

a) *$C = C_G(\phi)$ is periodic.*
b) *There is a finitely generated nilpotent subgroup N of G contained in $G\gamma$ such that G is periodic modulo N.*
c) *G is periodic modulo $G\gamma$.*
d) *G is periodic modulo* $\ker\psi$.

Proof: Suppose C is periodic. By [**4**, 3.17 and 3.13] there is a characteristic series
$$\langle 1\rangle = T_0 \leq T_1 \leq \cdots \leq T_s \leq T \leq T_\infty \leq G$$
of G, where each T_i/T_{i-1} is an abelian p'-group, T/T_s is a divisible abelian p-group of finite rank, T_∞/T is finite and T_∞ is the maximum periodic normal subgroup of G. Then Lemmas 5 and 6 yield that $C_{G/T}(\phi)$ is periodic.

Now G/T is (torsion-free)-by-finite ([**8**, Lemmas 4 and 6] again). Thus $C_{G/T}(\phi)$ is actually finite and therefore (see [**1**, Theorem D]) G has a ϕ-invariant normal subgroup H of finite index with $T \leq H$ and H/T nilpotent. Since we have assumed that G is countable, it follows from Part b) of the Proposition of [**11**] that H has a nilpotent subgroup K with H periodic modulo K (note we cannot in general choose K normal in H). Since ϕ has finite order we may choose K to be ϕ-invariant. Further K is periodic modulo some finitely generated subgroup L of K by Lemma 7. Again we may choose L to be ϕ-invariant. Since L is finitely generated and nilpotent, so $C_L(\phi)$ is finite. Consequently, e.g. by Theorem 1, there is a subgroup N of L of finite index with $L\gamma \supseteq N$. Clearly N is finitely generated and nilpotent, G is periodic modulo N and $G\gamma \supseteq L\gamma \supseteq N$. Therefore a) implies b). Trivially b) implies c), always c) implies d), and d) implies a) by Lemma 3. The proof is complete. □

The Proof of Theorem 2: Finite extensions of soluble FAR groups are always countable and satisfy min-q for every prime q. Thus Theorem 2 follows from Proposition 2. □

The Proof of the Corollary: Suppose $C_G(\phi)$ is periodic. Since ϕ has finite order, so G is the union of its ϕ-invariant subgroups X that are finite extensions of FAR groups. Then $C_X(\phi)$ is periodic and so X is periodic modulo $X\gamma$ by Theorem 2. Consequently if $x \in X$ then some positive power of x lies in $X\gamma$ and so in $G\gamma$. Thus a) implies b), b) implies c), and c) implies a) as in the previous cases. □

Lemma 8. *Let ϕ be an automorphism of a group G and A and B abelian ϕ-invariant normal subgroups of G with $B \leq A\gamma$. Set $C = C_G(\phi)$ and $K/B = C_{G/B}(\phi)$. Then $K = C(A \cap K)$. If C is finite and B has finite index in A, then K/B is finite.*

We will frequently apply Lemma 8 when A is central in G. In this case we may always choose $B = A\gamma$. If A is torsion-free of finite rank and C is periodic, then $A/A\gamma$ is finite by Lemma 1 and we can choose B of finite index in A.

Proof: Let $k \in K$. Then $k^{-1} \cdot k\phi \in B \leq A\gamma$, so $k^{-1} \cdot k\phi = a^{-1} \cdot a\phi$ for some $a \in A$, $(ka^{-1})\phi = ka^{-1}$, $k \in CA$ and $K = K \cap CA = C(A \cap K)$. If C and A/B are both finite, then clearly K/B is also finite. □

The proof of the following lemma is nearly a repeat of the proof of Lemma 4. As with Lemma 4 it suffices in Lemma 9 to assume that G is nilpotent of finite Hirsch number and satisfies min-q for some finite set π of primes q. However it is messy to specify π. Apart from the prime divisors of $|C|$ we also need certain primes that depend on the way ϕ acts on the torsion-free abelian ϕ-invariant sections of G. It is simpler to assume the FAR condition and hence to assume G satisfies min-q for all primes: q.

Lemma 9. *Let ϕ be an automorphism of the nilpotent FAR group G with $C = C_G(\phi)$ finite. Then $G\gamma$ has finite index in G.*

Proof: Assume by induction on the Hirsch number that the lemma is valid for groups (if any) of smaller Hirsch number than that of G. Let π denote the set of prime divisors of the order of C.

Consider first the case where the maximum normal locally finite subgroup T of G is finite. If G has Hirsch number 0 then $G = T$ is finite in this case and the claim is vacuous. Suppose G has positive Hirsch number. Then G is (torsion-free)-by-finite (e.g. by [**8**, Lemmas 4 and 6]). Also G is periodic and hence finite if its centre $\zeta_1(G)$ is finite (e.g. by Lemma 2), so G has an infinite, ϕ-invariant, torsion-free central subgroup Z. Then $C_Z(\phi) = C \cap Z = \langle 1 \rangle$ and $(Z : Z\gamma)$ is finite by Lemma 1. Since $Z\gamma$ is infinite and torsion-free, the Hirsch number of $G/Z\gamma$ is less than that of G. Further $C_{G/Z\gamma}(\phi)$ is finite by Lemma 8. By induction there exists a normal subgroup N of G of finite index with $Z\gamma \leq N$ and $N/Z\gamma$ lying in

$$(G/Z\gamma)\gamma = G\gamma \cdot Z\gamma/Z\gamma = G\gamma/Z\gamma,$$

since if $g \in G$ and $z \in Z$, then $g^{-1} \cdot g\phi \cdot z^{-1} \cdot z\phi = (gz)^{-1}(gz)\phi \in G\gamma$ by the centrality of z. Therefore $G\gamma \supseteq N$ and $G\gamma$ has finite index in G.

Now we consider the case where $O_\pi(G)$ is finite. We further induct on the least c such that $O_{\pi'}(G) \le \zeta_c(G)$. Obviously c is bounded by the class of G and the case $c = 0$ is covered by the case completed above. Suppose $c \ge 1$ and set $Z = O_{\pi'}(G) \cap \zeta_1(G) = O_{\pi'}(\zeta_1(G))$. Then ϕ acts fixed-point freely on Z, so $Z = Z\gamma$, e.g. by [**3**, 10.1.1]. Again $C_{G/Z}(\phi)$ is a finite π-group by Lemma 8. Clearly

$$O_{\pi'}(G/Z) = O_{\pi'}(G)/Z \le \zeta_{c-1}(G/Z).$$

By induction there is a normal subgroup N of G of finite index with $Z \le N$ and with N/Z contained in $(G/Z)\gamma = G\gamma/Z\gamma$. Then $G\gamma \supseteq N$ and $G\gamma$ has finite index in G.

We now consider the general case. By hypothesis $O_\pi(G)$ satisfies the minimal condition on subgroups. Hence $O_\pi(G)$ is a Chernikov group and as such has a characteristic subgroup D of finite index that is a direct product of a finite number, say r, of Prüfer q-groups for the various primes q in π. Here we induct on r, the case $r = 0$ having been covered above.

Assume $r > 0$. By Lemma 2 the subgroup $Z = O_\pi(G) \cap \zeta_1(G)$ is infinite. Then Z contains a characteristic (in G) divisible subgroup E of finite index, $E\phi = E \le D$, $E \ne \langle 1 \rangle$ and D/E is a direct product of less than r Prüfer groups. Also $E\gamma$ is divisible and $\ker(\gamma|_E) = C_E(\phi)$ is finite, so $E \cong E\gamma$ and $E = E\gamma$. Hence $C_{G/E}(\phi)$ is a finite π-group by Lemma 8 again. By induction there is a normal subgroup N of G of finite index with $E \le N$ and N/E contained in $(G/E)\gamma$. It follows that $G\gamma \supseteq N$ and that $G\gamma$ has finite index in G. The proof of the lemma is complete. □

The Proof of Theorem 3: a) By Fitting's Lemma there is a ϕ-invariant nilpotent normal subgroup M of G of finite index, which clearly must also be an FAR group. Suppose $C_G(\phi)$ is finite. Then by Lemma 9 there is a subgroup N of M of finite index with $M\gamma \supseteq N$. Clearly then N has finite index in G and $G\gamma \supseteq M\gamma \supseteq N$. Thus $G\gamma$ has finite index in G.

b) Now suppose ϕ has finite order m. If $G\gamma$ has finite index in G always so does $\ker\psi$. If $\ker\psi$ has finite index in G, then $C_G(\phi)$ has finite exponent by Lemma 3. But $C_G(\phi)$ is a finite extension of a nilpotent FAR group. Consequently $C_G(\phi)$ is finite. □

Lemma 10. *Let ϕ be an automorphism of the periodic soluble FAR group G. Then every orbit of ϕ in G is finite. If also $C_G(\phi) = \langle 1 \rangle$, then $G = G\gamma$.*

Proof: We induct on the derived length of G; if $G = \langle 1 \rangle$ everything is vacuous. Suppose G' satisfies the lemma and let $x \in G$. The abelian group G/G' satisfies min-q for every prime q and hence has only finitely many elements of each order. Thus $x\phi^m \in xG'$ for some positive integer m, say $x\phi^m = xy$ where $y \in G'$.

By induction $y\phi^n = y$ for some positive integer n. Also $y \cdot y\phi^m \cdot y\phi^{2m} \cdots y\phi^{m(n-1)}$ has finite order, r say. Then

$$x\phi^{mnr} = xy \cdot y\phi^m \cdot y\phi^{2m} \cdots y\phi^{m(nr-1)} = x(y \cdot y\phi^m \cdots y\phi^{m(n-1)})^r = x.$$

Thus $x\langle\phi\rangle$ is finite and ϕ has finite orbits in G.

Suppose $C_G(\phi) = \langle 1 \rangle$. If $x \in G$, then $X = \langle x\phi^i : i = 1, 2, \dots \rangle$ is finite and $C_X(\phi) = \langle 1 \rangle$. Hence $X = X\gamma$ by [**3**, 10.1.1]. Consequently $G = G\gamma$. □

The Proof of Theorem 4: Note first that $C_S(\phi)$ is finite for every ϕ-invariant section S of G by [**9**, Theorem ii)]. Let π denote the set of prime divisors of the order of $C_G(\phi)$. Denote the maximum periodic soluble normal subgroup of G by T_0 and set $T = O_{\pi'}(T_0)$. Then $T = T\gamma$ by Lemma 10.

Now T_0/T is a Chernikov group by [**4**, 3.17 and 3.13] (of course here π is finite). Hence in the notation of [**5**] the group G/T is a finite extension of a soluble FATR group and therefore its Fitting subgroup H/T is nilpotent and G/H is abelian-by-finite, see [**5**, 5.2.2]. By Theorem 3 there is a subgroup M/T of H/T of finite index with $(H/T)\gamma \supseteq M/T$; that is, with $H\gamma \cdot T\gamma = H\gamma \cdot T \supseteq M$. Replacing M by $M^{(H:M)}T$ we may assume that M is characteristic in G. Then G/M is finite-by-abelian-by-finite and hence is nilpotent-by-finite. By Theorem 3 again there is a subgroup N/M of G/M of finite index with $(G/M)\gamma \supseteq N/M$. Consequently $G\gamma \cdot M \supseteq N$ and therefore $(G\gamma)^{[3]} \supseteq G\gamma \cdot H\gamma \cdot T\gamma \supseteq N$. The proof is complete. □

Proposition 3. *Let ϕ be an automorphism of the finite extension G of a soluble FATR group. If $G\gamma$ has finite index in G, then $C_G(\phi)$ is finite.*

Proof: Now G has a characteristic series $\langle 1 \rangle = G_0 \le G_1 \le G_2 \le \cdots \le G_n \le G$, where each G_i/G_{i-1} is either a torsion-free abelian group of finite rank or is a divisible abelian q-group of finite rank for some prime q and G/G_n is finite, cf. [**8**, Lemma 4]. We induct on n and assume that G/G_1 satisfies the proposition. To simplify notation set $A = G_1$. Then $K/A = C_{G/A}(\phi)$ is finite.

By hypothesis there is a subgroup N of G of finite index contained in $G\gamma$. Let T be a (necessarily finite) transversal of A to K and consider $a \in A \cap N$. Then $a = g\gamma$ for some g in G. Necessarily $g \in K$. Hence $g = tb$ for some $t \in T$ and $b \in A$. Now $t\gamma \in K\gamma$, which lies in the abelian group A, so

$$a = g\gamma = g^{-1} \cdot g\phi = b^{-1}(t^{-1} \cdot t\phi)b\phi = t\gamma \cdot b\gamma.$$

Therefore $\bigcup_{t \in T} t\gamma \cdot A\gamma \supseteq A \cap N$. But T and $(A : A \cap N)$ are finite. Consequently $(A : A\gamma)$ is finite.

If A is torsion-free of finite rank, then $\operatorname{rank} A = \operatorname{rank} A\gamma$ and $C_A(\phi) = \ker(\gamma|_A)$ is torsion-free of rank 0. Thus in this case $C_A(\phi) = \langle 1 \rangle$. Suppose A is a divisible abelian q-group of finite rank. Then $A/A\gamma$ is finite and divisible. Hence $A = A\gamma$ and so $C_A(\phi) = \ker(\gamma|_A)$ is finite. Either way $C_A(\phi)$ is finite. Trivially $C_G(\phi) \leq K$ and $C_A(\phi) = A \cap C_G(\phi)$. Therefore $C_G(\phi)$ has order at most $|C_A(\phi)|(K : A)$, which is finite. □

Theorem 3 and Proposition 3 immediately yield the following.

Corollary. *Let ϕ be an automorphism of the finite extension G of a nilpotent FATR group. The following are equivalent.*

a) *$C_G(\phi)$ is finite.*
b) *$G\gamma$ has finite index in G.*

The Proof of Example 1: For any prime p let C be a Prüfer p-group and x the automorphism of C given by $c^x = c^{1+p}$ for all $c \in C$. Then x has infinite order. Let $A = C_1 \times C_2$ be the direct product of two copies of C and y the automorphism of A that acts as x on C_1 and as x^{-1} on C_2. The split extension $G = \langle y \rangle A$ of A by $\langle y \rangle$ is metabelian, minimax and of rank 3.

Now G has an automorphism ϕ of order 2 given by $y\phi = y^{-1}$ and $(c_1 d_2)\phi = d_1 c_2$ (here $c, d \in C$ with c_i and d_i denoting the corresponding elements of C_i). Then ϕ acts fixed-point freely on G/A and therefore $C_G(\phi) \leq A$, which is periodic.

Always $\ker \psi \supseteq G\gamma$. Suppose N is a normal subgroup of G with G/N periodic and $\ker \psi \supseteq N$. Then $y^i \in N$ for some positive integer i and hence $c_1^{-1} y^i c_1 \in N$ for every $c \in C$. But

$$c_1^{-1} y^i c_1 = y^i c_1^{-1} [c_1^{-1}, y^i] c_1 = y^i [c^{-1}, x^i]_1.$$

For any $i \geq 1$ we can pick $c \in C$ such that $[c^{-1}, x^i] \neq 1$. Then for this c we have

$$(c_1^{-1} y^i c_1)\psi = (y^i [c^{-1}, x^i]_1) \cdot y^{-i} [c^{-1}, x^i]_2) \neq 1.$$

Thus no such N exists. □

Remark. We know from Theorem 2 that in Example 1 there is some non-normal subgroup N of G with $G\gamma \supseteq N$ and G periodic modulo N. Such an N is very easy to find; in fact, it is elementary to check that $N = \langle y^2 \rangle$ suffices.

References

[1] E. Bettio, E. Jabara, and B. A. F. Wehrfritz, Groups admitting an automorphism of prime order with finite centralizer, Preprint (2011).

[2] G. Endimioni and P. Moravec, On the centralizer and the commutator subgroup of an automorphism, *Monatsh. Math.* **167(2)** (2012), 165–174. DOI: 10.1007/s00605-011-0298-0.

[3] D. Gorenstein, *"Finite groups"*, Harper & Row, Publishers, New York-London, 1968.

[4] O. H. Kegel and B. A. F. Wehrfritz, *"Locally finite groups"*, North-Holland Mathematical Library **3**, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[5] J. C. Lennox and D. J. S. Robinson, *"The theory of infinite soluble groups"*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2004. DOI: 10.1093/acprof:oso/9780198507284.001.0001.

[6] D. J. S. Robinson, *"Finiteness conditions and generalized soluble groups"*, Parts 1 and 2, Ergebnisse der Mathematik und ihrer Grenzgebiete **62** and **63**, Springer-Verlag, New York-Berlin, 1972.

[7] B. A. F. Wehrfritz, *"Group and ring theoretic properties of polycyclic groups"*, Algebra and Applications **10**, Springer-Verlag London, Ltd., London, 2009. DOI: 10.1007/978-1-84882-941-1.

[8] B. A. F. Wehrfritz, Almost fixed-point-free automorphisms of prime order, *Cent. Eur. J. Math.* **9(3)** (2011), 616–626. DOI: 10.2478/s11533-011-0017-z.

[9] B. A. F. Wehrfritz, On the fixed-point set and commutator subgroup of an automorphism of a group of finite rank, *Rend. Semin. Mat. Univ. Padova* **127** (2012), 249–255.

[10] B. A. F. Wehrfritz, On the fixed-point set and commutator subgroup of an automorphism of a soluble group, *Acta Math. Vietnam.* **37(3)** (2012), 419–426.

[11] B. A. F. Wehrfritz, Nilpotent-by-periodic commutator subgroups of automorphisms, *RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* (to appear).

School of Mathematical Sciences
Queen Mary University of London
Mile End Road
London E1 4NS
England
E-mail address: `b.a.f.wehrfritz@qmul.ac.uk`

Primera versió rebuda el 7 de març de 2012,
darrera versió rebuda el 2 d'agost de 2012.